Topology of Steady and Unsteady Incompressible Three-Dimensional Separated Flows

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§ 1. INTRODUCTION

Three-dimensional separated flow represents a domain of fluid mechanics of great practical interest that is now just beyond the reach of definitive theoretical analysis or numerical computation. It has been a topic of intensive study over the past three decades. Reliable theoretical analysis and numerical computation and proper interpretation of experimental observations all depend crucially on a correct understanding of the behavior of flow separation.

Steady three-dimensional flow separation has been studied by Maskell (1955), Legendre (1956, 1965, 1972, 1977, 1982), Werlé (1962, 1979), Lighthill (1963), Wang (1972, 1974, 1976), Perry and Fairlie (1974), Hsieh and Wang (1976), Hunt et. al. (1978), Han & Patel (1979), Tobak & Peake (1979, 1980, 1982), Dallmann (1983), Hornung & Perry (1984), and Zhang (1985). Important advances in the understanding of the nature of three-dimensional flow separation have been made which are well summarized in a recent review paper by Chapman (1986). In particular, it is now established (Lighthill, 1963) that the line of separation is itself a skin-friction line onto which adjacent skin-friction lines converge asymptotically, and is not an envelope of skin-friction lines as posed by Maskell (1955). However, issues concerning the origin of three-dimensional flow separation, and especially the existence of Wang's "open" separation, have not been completely resolved.

Unsteady flow separation, on the other hand, is not well understood, particularly in three dimensions. The often-quoted MRS (Moore 1958, Rott 1956, Sears 1956) criterion seems supported by some numerical computations and the analytic solution of Williams and Johnson (1974) to the unsteady boundary layer equations, but is difficult to apply in practice as the movement of the separation point is not known à priori. Virtually all theoretical studies on unsteady flow separation (e.g., Cebeci, 1982) are based on the two-dimensional boundary-layer equations. These cease to be valid at the onset of separation, so that no conclusion about the subsequent behaviour of flow separation can be drawn from them.

It is clear that a correct theory of the onset of flow separation and of the subsequent separated flow must be based on the full Navier-Stokes equations and not the boundary-layer equations. In this respect, it is interesting to note that whilst the solutions to the boundary layer equations may possess singularities (e.g. the Goldstein singularity on the line of separation), solutions of the Navier-Stokes equations are analytic everywhere. Therefore, it is not only more desirable but actually conceptually simpler to base our study of flow separation on the properties of solutions to the full Navier-Stokes equations rather than their boundary-layer approximation.

In this paper we shall restrict the scope of our investigation to the topological aspects of unsteady three-dimensional separated flows. In this regard, we observe that all results concerning the topology of steady three-dimensional separated flows that have been reported to date are obtainable solely on the basis of the following properties of the velocity field $\vec{V}$, namely

(i) $\vec{V}$ is analytic (1)

(ii) $$\nabla \cdot \vec{V} = 0 \qquad (2)$$

(iii) $\vec{V} = 0$ on the body surface (3)

Equation (3) expresses the no-slip boundary condition of the viscous fluid at the wall, whereas equation (2) is the continuity equation of an incompressible flow. * On the other hand, as pointed out earlier, the analyticity condition (1) of $\vec{V}$ is a property of any solution to the Navier-Stokes equations; $\vec{V}$ would be singular (e.g. the Goldstein singularity at the line of separation) if the flow were governed by the boundary-layer equations. These properties are shared, of course, by all solutions of the Navier-Stokes equations. On the other hand, separation properties that result from them are shared by classes of solutions.

It is the existence of properties that are shared by classes of solutions that suggests the adoption of a topological description of the flow, since topological properties also are shared by classes of solutions. By restricting our attention to topological properties only, we are able to avoid invoking the momentum equation which would be needed if we were to ask for the solution corresponding to specific boundary conditions. Nevertheless, the literature cited has shown that a fairly complete (albeit non-specific) topological description of separation in steady three-dimensional flow can be drawn based on Eqs (1) - (3) alone. In sections 2 and 3 we shall construct a mathematical framework for steady three-dimensional flow separation, ending up with a precise description of the class of flow separation we consider. The same framework will permit us, in section 4, to draw an analogous description of unsteady three-dimensional separated flow of the same class.

* The case of steady compressible flow is discussed in section 6.

We note that the analysis given in this paper based on the postulates (1) - (3) is actually more generally applicable than to the Navier-Stokes equations alone. In particular, the analysis will be applicable to whatever modeled equations are used to represent steady turbulent flow.

§ 2. CLASSIFICATION OF STEADY FLOW SEPARATION

Consider a steady flow of an incompressible viscous fluid over a body whose surface is B. Let $\xi\eta\varsigma$ be a local orthogonal curvilinear coordinate system such that $\varsigma = 0$ coincides with the body surface and the η-axis is along the line of separation (to be defined more precisely in § 2.2 and § 3.5). Denote the scale coefficients by $h_1(\xi,\eta,\varsigma)$, $h_2(\xi,\eta,\varsigma)$ and 1, respectively, and the corresponding unit vectors by $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$.

2.1 Properties of a Vector Field

Let $\vec{U}(\xi,\eta,\varsigma)$ be a vector field in the three-dimensional space $\xi\eta\varsigma$, where $\vec{U}$ is analytic jointly in ξ, η and ς. A field line of U is a curve whose tangent is everywhere parallel to $\vec{U}$, whereas a field surface is one whose normal is everywhere perpendicular to U. When the vector field is the flow velocity, its field lines are called streamlines and its field surfaces are called stream surfaces.

By the theorems of existence and uniqueness of solutions of ordinary differential equations, it is shown that through each regular point, where $\vec{U} \neq 0$, there passes one and only one field line. Consequently, if two field lines intersect with or are tangent to each other, the point of intersection or of tangency must be a singular point of the vector field where $\vec{U} = 0$. Moreover, a field line cannot end except at a singular point. On the other hand, there exist two independent families of field surfaces whose normals, while both perpendicular to $\vec{U}$, are different. Through a regular line on which $\vec{U} \neq 0$ (except possibly at a finite number of isolated points), there passes one and only one field surface of each family. Consequently, no two field surfaces of the same family can intersect with or be tangent to each other except along a singular line on which $\vec{U} = 0$. Also the boundary of a field surface, if it exists, must be a singular line. Furthermore, if two field surfaces of different families intersect with each other, the line of intersection must be a field line. On the other hand, they cannot be tangent to each other except along a singular line, as they have different normals.

To sum up: (a) the field lines and field surfaces are determined solely by the direction, and not the magnitude, of the vector field; (b) if two field lines intersect with or are tangent to each other the point of intersection or of tangency must be a singular point; (c) if two field surfaces intersect with each other, the line of intersection must be a field line; (d) if two field surfaces are tangent to each other, the line of tangency must be a singular field line.

In application to fluid flow we note that a stream surface must either originate from (or terminate at) infinity, or else, it must originate from (or terminate at) the body surface which, according to (3), is a singular field surface. This latter case is relevant in the study of flow separation.

We now compare the properties of the velocity field $\vec{V}$ and that of $\partial\vec{V}/\partial\varsigma$ near the body surface. The body surface is a singular surface of $\vec{V}$ according to Eq. (3), but is not, in general, a singular surface of $\partial\vec{V}/\partial\varsigma$. By its analyticity $\vec{V}$ is proportional to ς as $\varsigma \to 0$, hence the direction of $\vec{V}$ is the same as that of $\vec{V}/\varsigma$ as $\varsigma \to 0$. By L'Hospital's rule, $\vec{V}/\varsigma \to \partial\vec{V}/\partial\varsigma$ as $\varsigma \to 0$. So the direction of $\vec{V}$ is the same as the direction of $\dfrac{\partial\vec{V}}{\partial\varsigma}$ as $\varsigma \to 0$. Consequently, they have the same field lines and field surfaces near the body surface. Since $\left(\dfrac{\partial\vec{V}}{\partial\varsigma}\right)_{\varsigma=0}$ is proportional to the skin-friction vector $\vec{\epsilon}_w$, the limiting streamlines coincide with the skin-friction lines. As the magnitude of a vector field does not affect its field lines or field surfaces, we simply define

$$\vec{\epsilon}_w = \left(\frac{\partial\vec{V}}{\partial\varsigma}\right)_{\varsigma=0} . \tag{4}$$

A field surface of $\dfrac{\partial\vec{V}}{\partial\varsigma}$ and that of $\vec{V}$ are said to be adjunct field surfaces if they intersect with the body surface at the same line. Evidently, two adjunct field surfaces are tangent to each other, and hence have the same normal along their line of intersection with the body surface.

Furthermore, from the continuity equation (2), we have

$$\frac{\partial(h_2 u)}{\partial\xi} + \frac{\partial(h_1 v)}{\partial\eta} + \frac{\partial(h_1 h_2 w)}{\partial\varsigma} = 0 \tag{5}$$

where

$$\vec{V} = u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 \tag{6}$$

On the body surface, where (3) holds, Eq (5) reduces to

$$\left(\frac{\partial w}{\partial\varsigma}\right)_{\varsigma=0} = 0 \tag{7}$$

This shows that $\left(\dfrac{\partial\vec{V}}{\partial\varsigma}\cdot\vec{e}_3\right)_{\varsigma=0} = 0$, i.e.

$$\left(\frac{\partial\vec{V}}{\partial\varsigma}\cdot\vec{n}_B\right)_{\varsigma=0} = 0 \tag{8}$$

where $\vec{n}_B$ is the normal to the body surface. Accordingly, the body surface is a field surface of the vector field $\dfrac{\partial\vec{V}}{\partial\varsigma}$, and it must therefore also be a limiting stream surface of $\vec{V}$.

We note here that in what follows only the continuity equation on the body surface (7) is needed, but we do not need the full continuity equation (5). In other words, all the topological properties given in sections 2-3 are derived based on (1), (3) and (7). This point is important and will be used in extending (in section 6) the incompressible flow analysis to the compressible flow case.

2.2 Classification of Flow Separation

In what follows shall define flow separation in a way that will exclude consideration of what Wang (1974) has called "open" separation. This is not to deny the existence of such a category of flow but simply to affirm that it escapes our classification. A flow is said to separate from the body surface B if there exists a stream surface S that intersects B on the line Γ and if streamlines on S in the vicinity of Γ all originate from Γ and are directed away from Γ. We call S a separation stream surface and Γ a line of separation; the latter will be taken to be the η-axis. Flow attachment differs from flow separation merely in having an opposite flow direction, but otherwise has identical topological properties. For simplicity we shall refer, wherever no confusion may arise, only to flow separation with the understanding that whatever we say can be made to apply to flow attachment as well by a suitable reversal of flow directions.

Two mutually exclusive cases exist:

(1) The separation stream surface S is tangent to the body surface B along the whole of the separation line Γ. In this case the adjunct field surface of $\frac{\partial V}{\partial \varsigma}$ must also be tangent to the body surface along the same separation line Γ. This is possible only if the separation line is itself a singular line of the vector field $\left(\frac{\partial \vec{V}}{\partial \varsigma}\right)_{\varsigma=0}$, i.e. a singular line of the skin-friction vector field $\vec{\epsilon}_w$. This type of separation will be called singular tangent separation.

(2) The separation stream surface S intersects with the body surface B non-tangentially, i.e. at a non-zero finite angle along the line of separation Γ. In this case the adjunct separation field surface of $\frac{\partial V}{\partial \varsigma}$ also intersects with the body surface at non-zero angle along the same separation line Γ. Since the body surface is shown to be a field surface of $\frac{\partial V}{\partial \varsigma}$, the line of separation must be a field line of $\frac{\partial \vec{V}}{\partial \varsigma}$, and hence is itself a skin-friction line. Due to the analyticity of $\vec{\epsilon}_w$ the line of separation Γ must either be a singular skin-friction line, along which $\vec{\epsilon}_w = 0$ everywhere, or a regular skin-friction line containing, possibly, a finite number of isolated singular points of $\vec{\epsilon}_w$. In the former case the separation is called singular separation, whereas the latter case is called regular separation.

To sum up, within our classification, there exist two and only two types of flow separation of an incompressible viscous fluid:

(a) Regular separation, where the line of separation is itself a regular skin-friction line (containing, possibly, a finite number of singular points), from which the separation stream surface leaves the body surface at a non-zero angle.

(b) Singular separation, where the line of separation is a singular skin-friction line, from which the separation stream surface leaves the body surface either at a non-zero angle or tangentially along the line of separation.

Regular separation is the common type of flow separation in genuinely three-dimensional flow (Peake & Tobak, 1980) and will be studied in the next section. By contrast, two-dimensional and axisymmetric flow separation must be of singular type due to flow symmetry. If there exists a singular point of the skin-friction field from which a streamline leaves the body surface, symmetry requires that the singular point must lie on a singular line and the streamline must lie on a stream surface which leaves the body surface, rendering the separation singular.

As an example of tangent separation we cite the high Reynolds number flow past a slender body, e.g., a cone or a delta wing, at small incidence where the lines of separation are only slightly inclined to the direction of the main flow. F.T. Smith (1978) presented evidence showing that the limiting form of the flow at infinite Reynolds number is a potential flow in which are embedded vortex sheets carrying concentrated vorticity. He also showed that the vortex sheets must separate tangentially from the body surface. This type of flow separation at infinite Reynolds number thus belongs to the class of singular tangent separation. Tangent separation will be shown (in section 5) to prevail as well when flow separation first appears in the impulsively started flow past a circular cylinder.

§ 3. REGULAR SEPARATION

In this section we shall study the local behavior of the flow field near the line of separation.

3.1. Existence of a Singular Point on the Line of Separation

In regular separation, the separation stream surface S leaves the body surface B with a non-zero angle along the separation line Γ. Consequently

$$\left(\vec{n}_S \times \vec{e}_3\right)_\Gamma \neq 0 \tag{9}$$

where $\vec{n}_S$ is the unit normal of the separation stream surface.

Now, streamlines on the separation stream surface S in the vicinity of the separation line Γ originate from Γ. If all such streamlines on S intersect with Γ tangentially to the body surface B, then $\left(\vec{n}_S \times \vec{e}_3\right)_\Gamma = 0$ along the separation line Γ, contradicting (9). Hence, there must be at least one streamline on S that intersects with the body surface B at some point P on Γ, making a non-zero angle to the body surface. Let the equation of this streamline be given parametrically by $\xi = k_1(\tau)$, $\eta = k_2(\tau)$, $\varsigma = k_3(\tau)$. Then

$$\left(\frac{dk_3}{d\tau}\right)_P \neq 0 \tag{10}$$

Since the direction of this streamline at the point of intersection P is parallel to the limiting direction of $\vec{V}$ at P, which, in turn, is parallel to $\left(\frac{\partial \vec{V}}{\partial \varsigma}\right)_P$, we get

$$\frac{\partial u/\partial \varsigma}{dk_1/d\tau} = \frac{\partial v/\partial \varsigma}{dk_2/d\tau} = \frac{\partial w/\partial \varsigma}{dk_3/d\tau} \quad \text{at} \quad P \tag{11}$$

With condition (10) and $\left(\frac{\partial w}{\partial \varsigma}\right)_P = 0$ from (7), Eqs. (11) yield

$$\left(\frac{\partial u}{\partial \varsigma}\right)_P = \left(\frac{\partial v}{\partial \varsigma}\right)_P = 0 \tag{12}$$

So the point of intersection P is a singular point of the skin-friction field $\vec{\epsilon}_w$.

We conclude that there must exist at least one singular point of the skin-friction field $\vec{\epsilon}_w$ on the line of separation in regular separation. The above arguments also show that any streamline on the separation stream surface that intersects with the body surface must do so at a singular point on Γ. Such singular points are isolated on Γ in regular separation. They, together with the remaining singular points in the skin-friction field, must obey certain topological rules as described by Hunt, et. al. (1978). In particular, the number of nodal points must exceed the number of saddle points by two on any smooth body surface that is topologically equivalent to a sphere.

3.2. A Necessary Condition

As shown in § 2.2, the line of separation Γ is itself a skin-friction line. Its equation may then be given by

$$\frac{h_1 d\xi}{\left(\frac{\partial u}{\partial \varsigma}\right)_{\varsigma=0}} = \frac{h_2 d\eta}{\left(\frac{\partial v}{\partial \varsigma}\right)_{\varsigma=0}} \quad \text{on} \quad \Gamma \tag{13}$$

Since Γ is taken to be the η-axis, we also have

$$d\xi : d\eta = 0 : 1 \quad \text{on} \quad \Gamma \tag{14}$$

Therefore, from (13) and (14) we get

$$\left(\frac{\partial u}{\partial \varsigma}\right)_{\xi=\varsigma=0} = 0 \tag{15}$$

The above analysis shows that (15) is a necessary condition for the η -axis, (i.e. $\xi = \varsigma = 0$) to be a line of separation. However, contrary to Zhang's (1985) conclusion, it is not a sufficient condition for the η -axis to be a line of separation.

3.3 The Flow Reversal Condition

In the situation of flow separation, it is clear that the body surface B and the separation stream surface S constitute two barriers to the flow such that near and on opposite sides of the line of separation Γ a certain component of the flow must reverse direction. In particular, $(\frac{\partial u}{\partial \varsigma})_{\substack{\varsigma=0\\ \xi\to 0+}} < 0$ and $(\frac{\partial u}{\partial \varsigma})_{\substack{\varsigma=0\\ \xi\to 0-}} > 0$ for flow separation where fluid flows away from the body surface $\varsigma = 0$. Likewise, $(\frac{\partial u}{\partial \varsigma})_{\substack{\varsigma=0\\ \xi\to 0+}} > 0$ and $(\frac{\partial u}{\partial \varsigma})_{\substack{\varsigma=0\\ \xi\to 0-}} < 0$ for flow attachment where fluid flows toward the body surface. Consequently we obtain

$$R(\eta) \equiv \left(\frac{\partial^2 u}{\partial \xi \partial \varsigma}\right)_{\xi=\varsigma=0} < 0 \quad \text{for flow separation} \tag{16}$$

$$R(\eta) = \left(\frac{\partial^2 u}{\partial \xi \partial \varsigma}\right)_{\xi=\varsigma=0} > 0 \quad \text{for flow attachment} \tag{17}$$

These flow reversal conditions, which are direct generalizations of the conditions for two-dimensional flow, were first obtained by Zhang (1985).

3.4 Types of Singular Points on the Line of Separation

(1) Velocity field near a singular point

Without loss of generality, we let the singular point 0 on Γ be at $\xi = \eta = \varsigma = 0$. Expanding the velocity components u,v,w as Taylor series about the point 0 and using (3) yield

$$u = (a_1\, \xi + b_1\, \eta + c_1\varsigma)\varsigma + \cdots \tag{18a}$$

$$v = (a_2\, \xi + b_2\, \eta + c_2\varsigma)\varsigma + \cdots \tag{18b}$$

$$w = (a_3\, \xi + b_3\, \eta + c_3\, \varsigma)\varsigma + \cdots \tag{18c}$$

where a_1, ..., c_3 are constants, and "+..." denotes higher order terms in ξ, η, ς.

As (15) and (7) imply, respectively,

$$b_1 = 0\,, \tag{19a}$$

$$a_3 = b_3 = 0\,, \tag{19b}$$

the streamlines near the singular point 0(0,0,0) on the line of separation are therefore given by the following differential equations

$$\frac{h_{10}d\xi}{a_1\xi + c_1\varsigma} = \frac{h_{20}d\eta}{a_2\xi + b_2\eta + c_2\varsigma} = \frac{d\varsigma}{c_3\varsigma} \tag{20}$$

where $h_{i0} = (h_i)_{\xi=\eta=\varsigma=0}$, i=1,2 . We now investigate the behavior of the streamlines near the singular point, first on the body surface B and then on the separation stream surface S.

(2) On the body surface B

As the body surface is approached, $\varsigma \rightarrow 0$ and the streamlines coincide with the skin-friction lines as noted in section 2.1. The equations for the skin-friction lines are obtained from (20) with $\varsigma = 0$:

$$\frac{h_{10}d\xi}{h_{20}d\eta} = \frac{a_1\xi}{a_2\xi + b_2\eta} \tag{21}$$

According to singularity theory of ordinary differential equations and the fact that $h_{10} > 0$ and $h_{20} > 0$, the nature of the singular point 0 of the skin-friction field is determined by the sign of

$$q_B = a_1 b_2 \tag{22}$$

In particular, a node (including a focus) corresponds to $q_B > 0$, whereas a saddle point corresponds to $q_B < 0$.

(3) On the separation stream surface S

Let the equation of the separation stream surface be given by

$$S\colon\ \varsigma = F(\xi,\eta) \tag{23}$$

As S intersects the body surface $\varsigma = 0$ along the η-axis, we have

$$F(0,\eta) = 0 \tag{24}$$

Expanding $F(\xi,\eta)$ as a Taylor series about the point $\xi = \eta = 0$ and using (24) we get

$$\varsigma = F(\xi,\eta) = k\xi + 0(\xi^2,\xi\eta) \tag{25}$$

The equations of the streamlines on the separation stream surface S are obtained by substituting (25) in (20):

$$\frac{h_{10}d\xi}{(a_1 + kc_1)\xi} = \frac{h_{20}d\eta}{(a_2 + kc_2)\xi + b_2\eta} = \frac{d\xi}{c_3\xi} \tag{26}$$

Accordingly, we get

$$k = (h_{10}c_3 - a_1)/c_1 \tag{27}$$

and

$$\frac{1}{h_{20}} \frac{d\xi}{d\eta} = \frac{c_3\xi}{(a_2 + kc_2)\xi + b_2\eta} \tag{28a}$$

$$\frac{d\zeta}{d\xi} = k \tag{28b}$$

Equation (27) determines the local slope k of the separation stream surface S at the singular point 0, whereas Eqs (28) determine the streamlines on S. With $\xi\eta$ as the surface coordinates of S, singularity theory of ordinary differential equations again asserts that the nature of the singular point 0 of the flow field $\vec{V}$ on the separation stream surface is determined by the sign of

$$q_S = c_3 b_2 \tag{29}$$

In particular, a node (including a focus) corresponds to $q_S > 0$, whereas a saddle point corresponds to $q_S < 0$.

Now, in the case of flow separation we have $c_3 > 0$, but $a_1 = R(0) < 0$ from (16), so c_3 and a_1, and hence q_B and q_S, are of opposite signs. We conclude that a singular point of the flow field on the line of separation must be either

(a) a saddle point in the skin-friction field on the line Γ and, at the same location, a nodal point on the separation stream surface S. Because of the presence of the body surface B, the node on S is one-sided and hence can only be a regular node, not a focus. The streamlines on S all originate at the nodal point and are directed away from it. In effect, they have entered the stream surface through the saddle point in the skin-friction field on the line of separation. We call the saddle point a saddle point of separation. The flow is illustrated in Fig. 1.

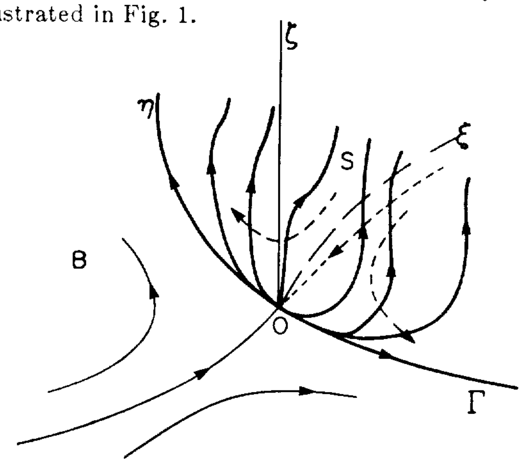

Fig 1. Flow near a saddle point of separation 0 of the skin-friction field on the line of separation Γ. With flow direction reversed this figure also represents the flow near a saddle point of attachment.

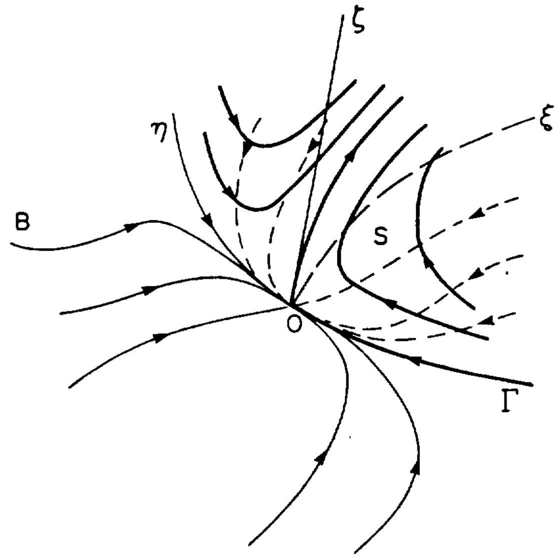

Fig 2a. Flow near a nodal point of separation 0 of the skin-friction field on the line of attachment Γ. With flow direction reversed this figure also represents the flow near a nodal point of attachment.

or

(b) a nodal point in the skin-friction field on the line Γ and, at the same location a half-saddle point on the separation stream surface S. In this case, there is only one streamline on S that has entered through the node in the skin-friction field. This node may be either a regular node or a focus, but in either case it must be a node of separation. The flow is illustrated in Fig. 2.

3.5 Distribution of Singular Points on the Line of Separation

The flow direction of the skin-friction field $\vec{\epsilon}_w$ along the line of separation, which is the η-axis, is determined by $\left(\dfrac{\partial v}{\partial \varsigma}\right)_{\varsigma=\xi=0} \equiv D(\eta)$. Near the singular point 0(0,0,0) we have, from (18b)

$$D(\eta) = b_2\eta + 0(\eta^2) \tag{30}$$

At a saddle point of separation $a_1 = R(0) < 0$ and $q_B = a_1 b_2 < 0$, hence $b_2 > 0$ and the flow is away from the saddle point. Similarly, at a nodal point of separation $a_1 < 0$ and $q_R = a_1 b_2 > 0$, hence $b_2 < 0$ and the flow is toward the nodal point.

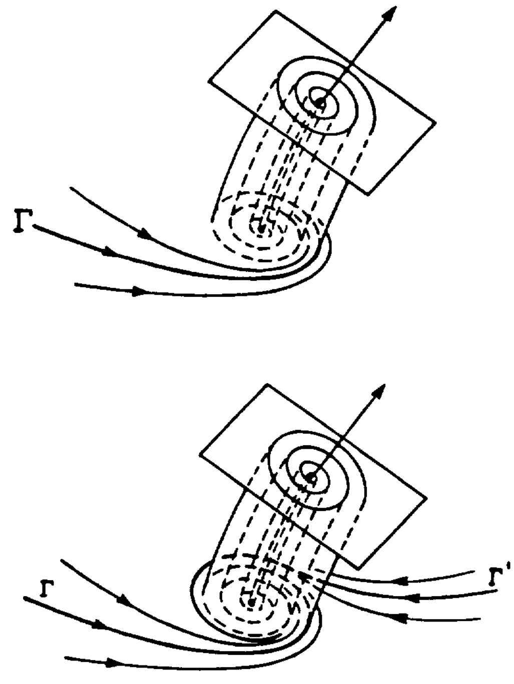

Fig 2b. Flow near a focus of separation of the skin-friction field.

Furthermore, on the line of separation Γ where $\xi = \varsigma = 0$, we have $\left(\dfrac{\partial u}{\partial \varsigma}\right)_\Gamma = 0$ from (15). Hence, every point P on Γ where

$$D(\eta)_{P \in \Gamma} = 0 \tag{31}$$

is a singular point of $\vec{\epsilon}_w$. Consequently, $D(\eta)$ changes sign only when passing a singular point on Γ.

Combining the above two results, we see that along the line of separation $\vec{\epsilon}_w$ must always flow from a saddle point of separation (S_s) toward a node of separation (N_s). (By reversing the direction of the flow field we see also that along a line of attachment $\vec{\epsilon}_w$ must always flow from a node of attachment (N_a) toward a saddle point of attachment (S_a).)

It is now evident that a line of separation Γ in regular separation must originate at a saddle point of separation. If Γ is closed it must also contain a node of separation, this being a consequence of the continuity of flow direction along Γ and of condition (31). On the other hand, if Γ is an open curve on the surface of a three-dimensional finite body its end-points must be singular points of $\vec{\epsilon}_w$. The rule governing the direction of flow along the line of separation then requires that these end-points be nodes of separation.

3.6 Summary of Properties of Regular Flow Separation

The mathematical properties of regular flow separation derived in sections 2 and 3 are summarized as follows.

(a) The line of separation Γ is itself a skin-friction line. Consequently, when it is chosen as the η-axis of a curvilinear orthogonal coordinates $\xi\eta\varsigma$ it is necessary that

$$\left(\frac{\partial u}{\partial \varsigma}\right)_{\xi=\varsigma=0} = 0 \tag{32}$$

Also

$$\left(\frac{\partial^2 u}{\partial \xi \partial \varsigma}\right)_{\xi=\varsigma=0} < 0 \quad \text{for flow separation} \tag{33a}$$

$$\left(\frac{\partial^2 u}{\partial \xi \partial \varsigma}\right)_{\xi=\varsigma=0} > 0 \quad \text{for flow attachment} \tag{33b}$$

(b) The line of separation must originate from a saddle point of the skin-friction field. It must end at a nodal point of separation if it is a closed curve or at a pair of nodal points of separation if it is an open curve.

(c) A saddle point of the skin-friction field on the line of separation is simultaneously a half-nodal point of the flow field on the separation stream surface (cf. Fig. 1). A nodal point of separation of the skin-friction field on the line of separation is simultaneously a half-saddle point of the flow field on the separation stream surface (cf. Fig. 2).

§ 4. UNSTEADY THREE-DIMENSIONAL FLOW SEPARATION

By unsteady flow separation we mean time-dependent flow separation relative to an observer on the body surface. An unsteady flow is said to separate from the body surface B at time t if there exists a stream surface S at t that intersects B on the line Γ and if the streamlines on S at time t in the vicinity of Γ all originate from Γ and are directed away from it. The stream surface S is called the instantaneous separation stream surface at the instant t, and Γ the instantaneous line of separation. Flow attachment at time t is defined analogously with flow directions reversed.

To describe unsteady flow separation relative to an observer on the body surface, it is imperative that we use a frame of reference that is fixed to the body surface. The body in question may be a rigid body performing a given motion, or it may be a deformable body.

Now for an unsteady incompressible flow viewed in a frame of reference fixed to the body surface, the continuity equation at any instant of time remains the same as that of steady flow, i.e.

$$\nabla \cdot \vec{V}(\vec{r};t) = 0 \tag{34}$$

Furthermore,

$$\vec{V}(\vec{r};t) \quad \text{is analytic in the spatial variables} \quad \vec{r} \tag{35}$$

being a consequence of the Navier-Stokes equations. With our choice of the frame of reference, the boundary condition that there be no slip at the body surface remains

$$\vec{V}(\vec{r};t) = 0 \quad \text{on the body surface} \tag{36}$$

even though the flow above the surface may be unsteady.

We note that the time variable t appears only as a parameter in Eqs. (34) - (36) which are otherwise identical to Eqs. (1) - (3). Of course, the time t appears as a genuine independent variable through the term $\frac{\partial \vec{V}}{\partial t}$ in the momentum equation. The time-history effects of the unsteady motion of the fluid are thus introduced only through the momentum equation, which now contains an inertia force term $\vec{F}_i$ arising from the motion, or deformation, of the body in addition to the external force $\vec{G}$. For instance,

$$\vec{F}_i = -\left[\frac{d\vec{V}_c}{dt} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{V}) + \frac{d\vec{\Omega}}{dt} \times \vec{r}\right] \tag{37}$$

for a rigid body where $\vec{V}_c$ is the velocity of the center of mass of the body and $\vec{\Omega}$ is its angular velocity.

As we have noted in § 1, the momentum equation for steady flow, which would be needed to determine the separated flow field uniquely, could be by-passed if we asked only for a topological description of the flow near the body surface. Analogously, we can avoid invoking the unsteady momentum equation by again asking only for a topological description of unsteady flow separation, this time based on Eqs (34) - (36) instead of Eqs. (1) - (3). By comparing the two sets of equations, we conclude that separation of an unsteady incompressible viscous flow at time t, when viewed from a frame of reference fixed to the body surface, is topologically the same as that of the fictitious steady flow obtained by freezing the unsteady flow at the instant t. In other words, the topological properties of unsteady flow separation at time t, as recorded by, say, a snapshot of the flow, are governed by the same rules that govern separation in steady flow. In this sense, all results in §§ 2 and 3 for steady separation apply to unsteady flow separation instantaneously.

We further remark that the MRS criterion of unsteady flow separation aims at answering the question of "massive" flow separation which is determined by the momentum equation with an inertia force term, whereas our theory based on equations (34) - (36) and ignoring the momentum equation describes only the local behavior of flow separation near the body surface. Of course, the behavior of flow separation as described in this paper must always be present locally in any "massively" separated flow field.

§ 5. APPLICATIONS

This section is devoted to a preliminary, qualitative discussion of some aspects of unsteady flow separation.

5.1 The Stream Functions

It is well known that the continuity equation (34) may be replaced by introducing two stream functions $\psi(\vec{r};t)$ and $\chi(\vec{r};t)$, namely

$$\vec{V} = \nabla\psi \times \nabla\chi \tag{38}$$

Evidently, Eq (34) is automatically satisfied, and $\vec{V}$ is analytic in $\vec{r}$ provided ψ and χ are. Also $\psi(\vec{r};t)$ = const. and $\chi(\vec{r};t)$ = const. represent two families of instantaneous stream surfaces at time t.

The equation of the body surface in the body surface-fixed coordinates must be independent of time, and hence must be of the form

$$B(\vec{r}) = 0 \tag{39}$$

As the body surface is a stream surface for all time, we choose the ψ-family of stream surfaces such that $\psi(\vec{r};t) = 0$ contains the body surface. To satisfy condition (36) it is then necessary and sufficient that ψ be of the form

$$\psi(\vec{r};t) = B^2(\vec{r})\ S(\vec{r};t) \tag{40}$$

where S and B are functionally independent. It is easily shown that the surface

$$S(\vec{r};t) = 0 \quad , \tag{41}$$

if it exists, is also a stream surface. Furthermore, if it intersects with the body surface at the instant t, it is an instantaneous separation stream surface and the intersection is an instantaneous line of separation. An important problem is to study the evolution of the surface $S(\vec{r};t) = 0$, especially near the time $t = T_S$ when it first intersects with the body surface.

5.2 Onset of Separation for Impulsively Started Flow past a Circular Cylinder

As an example of the problem just cited, consider the two-dimensional impulsive incompressible flow U_∞ past a stationary circular cylinder of radius a. For high Reynolds number flow, $R_e = \dfrac{U_\infty a}{\nu} >> 1$, C.Y. Wang (1967) used the method of matched asymptotic expansions to obtain a uniformly valid solution to the third order in $\epsilon = 1/R_e$ that is valid for small time. His solution for the stream function in polar coordinates (r,θ) is

$$\psi = \sin\theta\,(r - \frac{1}{r}) + \epsilon\, 4\sqrt{\frac{t}{\pi}}\sin\theta\left(-\frac{1}{r} + e^{-\eta^2} - \sqrt{\pi}\,\eta\, erfc\,\eta\right)$$

$$+ \epsilon^2 \frac{4t}{\sqrt{\pi}}\sin\theta\left[-\frac{\sqrt{\pi}}{4}\, erf\,\eta + \frac{3\sqrt{\pi}}{2}\eta^2\, erfc\,\eta - \frac{3}{2}\eta\, e^{-\eta^2}\right]$$

$$+ \epsilon^2\, 8t^{3/2}\sin\theta\cos\theta\, G_3(\eta) + 0(\epsilon^3) \tag{42}$$

where all the lengths are measured in units of a, velocity components in units of U_∞. Hence ψ is measured in units of $U_\infty a$ and we choose to measure time t in units of $\epsilon a/U_\infty$. The variable η is related to r by $\eta = (r-1)/(2\epsilon\sqrt{t})$. The first term in (42) represents the inviscid potential flow at the initial instant, $t = 0$. The solution (42) may also be written

$$\psi(\eta,\theta;t) = 4\epsilon\sqrt{t}\sin\theta\left[G_1(\eta) + \epsilon\sqrt{t}\{G_2(\eta) + 2\sqrt{t}\, G_3(\eta)\cos\theta\}\right] \tag{43}$$

where

$$G_1(\eta) = \eta\, erf\,\eta + \frac{1}{\sqrt{\pi}}(e^{-\eta^2} - 1) \tag{44}$$

$$G_2(\eta) = -\eta^2 + \frac{2}{\sqrt{\pi}}\eta - \frac{1}{4}\, erf\,\eta + \frac{3}{2}\eta^2\, erfc\,\eta - \frac{3}{2\sqrt{\pi}}\eta\, e^{-\eta^2} \tag{45}$$

$$G_3(\eta) = \frac{11}{6\sqrt{\pi}} e^{-\eta^2} erfc\,\eta - \frac{8}{3\sqrt{2\pi}}\, erfc\,\sqrt{2}\,\eta + \frac{\eta^3}{3}\, erfc^2\,\eta$$

$$- \frac{2}{3\sqrt{\pi}}\eta^2 e^{-\eta^2}\, erfc\,\eta + \frac{1}{3\pi}\eta e^{-2\eta^2} - \frac{\eta}{2}\, erfc^2\,\eta$$

$$+ \frac{1}{\sqrt{\pi}}\left(\frac{4}{9\pi} - \frac{3}{2}\right)e^{-\eta^2} - \left(1 + \frac{4}{9\pi}\right)\eta^3\, erfc\,\eta$$

$$+ \frac{1}{\sqrt{\pi}}\left(1 + \frac{4}{9\pi}\right)\eta^2 e^{-\eta^2} + \frac{2}{3\sqrt{\pi}}\, erfc\,\eta + \left(\frac{1}{2} - \frac{2}{3\pi}\right)\eta\, erfc\,\eta$$

$$+ \frac{1}{\sqrt{\pi}}\left(\frac{8}{3\sqrt{2}} - \frac{4}{9\pi} - 1\right). \tag{46}$$

Here erf is the error function, $erf\,\eta = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\varsigma^2}d\varsigma$, and the complementary error function $erfc\,\eta = 1 - erf\,\eta$. It can easily be shown that at the body surface, $\eta = 0$, $G_i(0) = G'_i(0) = 0$, $i = 1,2,3$. Hence (43) is of the form

$$\psi = 4\,\epsilon \sqrt{t}\,\sin\theta\;\eta^2\, S(\eta,\theta;t) \tag{47}$$

in conformity with Eq (36). In Eq (47)

$$S(\eta,\theta;t) = g_1(\eta) + \epsilon\sqrt{t}\left[g_2(\eta) + 2\sqrt{t}\; g_3(\eta)\cos\theta\right] \tag{48}$$

where $g_i(\eta) = G_i(\eta)/\eta^2$, $i = 1,2,3$. The functions $g_i(\eta)$ are plotted in Fig. 3. In particular, near the body surface, i.e. for η small, we get

$$g_1(\eta) = \frac{1}{\sqrt{\pi}} + 0(\eta^2)$$

$$g_2(\eta) = \frac{1}{2} - \frac{4}{3\sqrt{\pi}}\,\eta + 0(\eta^2)$$

$$g_3(\eta) = \frac{1}{\sqrt{\pi}}\,(1 + \frac{4}{3\pi}) - \frac{2}{3}\,\eta + 0(\eta^2)$$

Hence

$$S(\eta,\theta;t) = \frac{1}{\sqrt{\pi}}\left[\left\{1 + \epsilon\sqrt{t}\left(\frac{\sqrt{\pi}}{2} - \frac{4}{3}\,\eta\right)\right\} + 2\epsilon t\left(1 + \frac{4}{3\pi} - \frac{2}{3}\,\eta\right)\cos\theta\right] \tag{49}$$

The function S given by (49) is now used to study the behavior of flow separation near the body surface.

Although Wang's analysis is based on the assumption of small time, the solution may be used to give qualitative results for larger times. For this purpose we let time T be measured in units of a/U_∞, i.e. $T = \epsilon t$. Eq. (49) then becomes

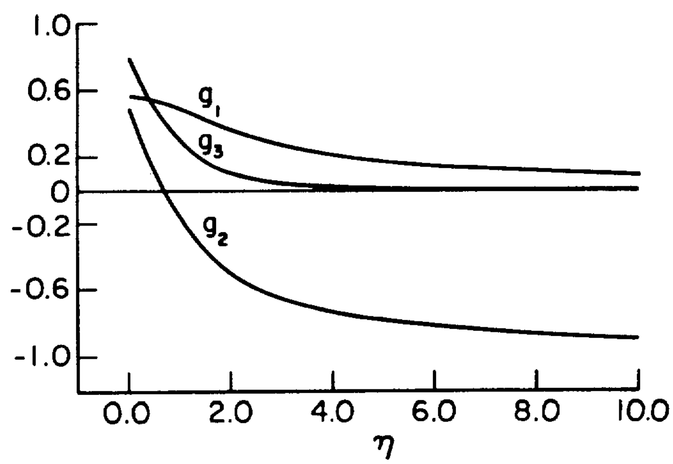

Fig 3. The functions $g_1(\eta)$, $g_2(\eta)$, $g_3(\eta)$.

$$S(\eta,\theta;T) = \frac{1}{\sqrt{\pi}} \left[\left\{ 1 + \sqrt{\epsilon T} \left(\frac{\sqrt{\pi}}{2} - \frac{4\eta}{3} \right) \right\} + 2T \left(1 + \frac{4}{3\pi} - \frac{2\eta}{3} \right) \cos\theta \right] \tag{50}$$

We observe from (50) that

(a) Only after a finite time can it happen that $S = 0$.

(b) The time T when the surface $S = 0$ intersects with the body surface $\eta = 0$ is given by

$$2 + \sqrt{\epsilon \pi T} + 4(1 + \frac{4}{3\pi})T \cos\theta = 0 \tag{51}$$

It is clear from (51) that the surface $S = 0$ will first intersect with the body surface at $\theta = \pi$, i.e. at the rear stagnation point.

(c) The separation time T_s, defined as the time when the surface $S = 0$ first intersects with the body surface, is given by

$$T_s = \left[\frac{\sqrt{32(1 + \frac{4}{3\pi}) + \epsilon\pi} + \sqrt{\epsilon\pi}}{8(1 + \frac{4}{3\pi})} \right]^2$$

$$= \frac{1}{2(1 + \frac{4}{3\pi})} + 0(\epsilon^{1/2})$$

$$= 0.35 + 0(\epsilon^{1/2}) \tag{52}$$

We note that the separation time T_s calculated above as the time when the separation surface first appears is identical to Wang's estimate of the time when the surface shear stress first becomes zero.

(d) From (50) we see that at the separation time T_s, whence $\theta = \pi$,

$$\left(\frac{\partial \eta}{\partial \theta} \right)_{\substack{S=0 \\ \eta=0 \\ T=T_s}} = 0 \tag{53}$$

This shows that initially the separation surface $S = 0$ leaves the body surface tangentially, i.e., initial separation is a tangent singular separation. However, at any later time $T > T_s$ it is a non-tangential singular separation as

$$\left(\frac{\partial \eta}{\partial \theta} \right)_{\substack{S=0 \\ \eta=0 \\ T>T_s}} \neq 0 \tag{54}$$

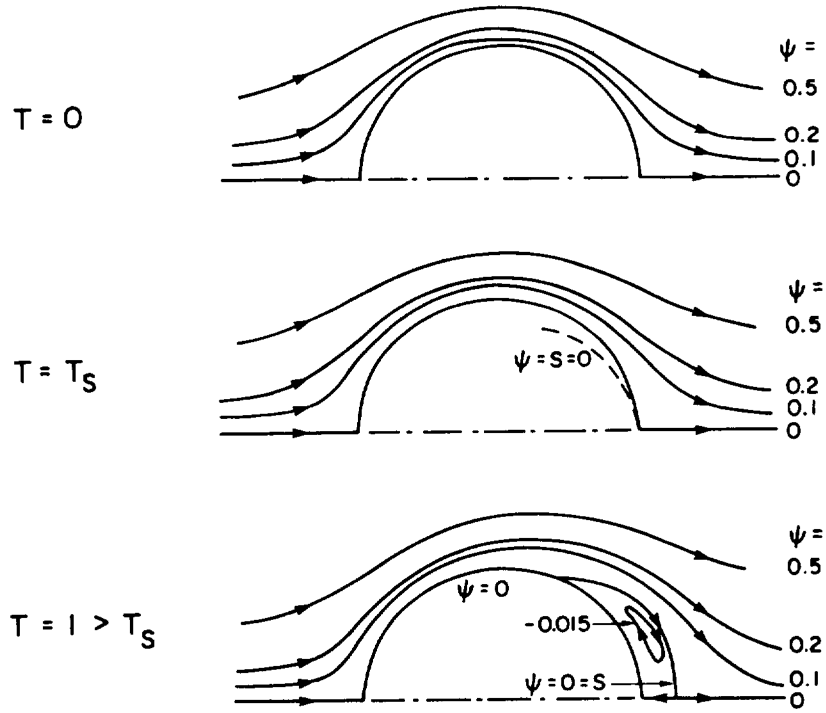

Fig 4. Impulsive flow past a stationary cylinder, $R_e = 100$.

In Fig 4 are shown the stream surfaces, including $S = 0$, at three different times $T = 0$, $T = T_s$ and $T = 1 > T_s$, all cases for $R_e = 100$. It is seen that the surface $S = 0$ emerges from inside the cylinder; as soon as it moves into the flow field, the flow becomes separated.

Impulsive flow past a stationary sphere can be studied similarly by taking

$$\begin{cases} \psi = \psi(r,\theta;t) \\ \chi = \phi \end{cases} \tag{55}$$

for symmetric flow, where (r,θ,ϕ) are spherical coordinates. The flow separation properties are expected to be qualitatively the same as those for the impulsive flow past a circular cylinder: at some finite time T_s after starting, a separation stream surface will first appear tangentially at the rear stagnation point; it will grow with time, immediately becoming non-tangential to the body surface as it emerges into the flow field.

5.3 Effects of Expansion of a Circular Cylinder on Flow Separation

An interesting numerical study has just been made by Lin, Mekala, Chapman and Tobak (1986) on the migration of the separation point on a deforming cylinder. The qualitative aspects of the effects of deformation of the cylinder on the onset of flow separation can now be discussed from a frame of reference fixed to the body surface. From this frame of reference the effect of acceleration and of deformation of the cylinder surface is equivalent to adding appropriate inertia forces.

To consider the effect of surface deformation alone on flow separation, let $a(t)$ be the radius of the cylinder and r the distance of a fluid particle from the surface of the cylinder (Fig. 5). Let $\hat{r}$ and $\hat{\theta}$ be unit vectors in the ratial and azimuthal directions respectively. The absolute acceleration of the fluid particle is

$$\vec{A} = (\ddot{a} + \ddot{r})\hat{r} + 2(\dot{a} + \dot{r})\dot{\theta}\hat{\theta} - (a + r)\dot{\theta}^2\hat{r} + (a + r)\ddot{\theta}\hat{\theta}$$

$$= \left[\left\{\ddot{r} - (a + r)\dot{\theta}^2\right\}\hat{r} + \left\{(a + r)\ddot{\theta} + 2\dot{r}\dot{\theta}\right\}\hat{\theta}\right] + \left[\ddot{a}\hat{r} + 2\dot{a}\dot{\theta}\hat{\theta}\right]$$

$$\equiv \vec{A}_0 + \vec{A}_D \tag{56}$$

where $\vec{A}_0$, the quantity in large brackets, is the acceleration of the particle that would be present alone if there were no deformation of the cylinder, and $\vec{A}_D$ is the additional acceleration arising from the deformation. Relative to the cylinder surface at time t the inertia force $\vec{F}_i$ of a particle of unit mass is thus

$$\vec{F}_i = -\vec{A}_D = -\ddot{a}\hat{r} - 2\dot{a}\dot{\theta}\hat{\theta} \tag{57}$$

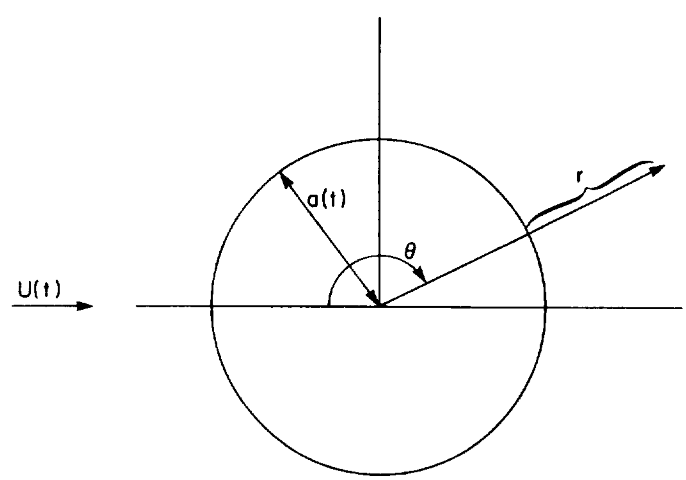

Fig 5. Uniform flow past a deforming circular cylinder showing notation.

In the case of a flow U past a circular cylinder whose surface is expanding at constant rate (Fig. 5) $\dot{a} > 0$, $\ddot{a} = 0$, the inertia force acting on the particle is equal to $-2\dot{a}\dot{\theta}\hat{\theta}$. This force acts in the direction just opposite to the motion ($\dot{\theta} > 0$) of the particles near the body surface, and, like an adverse pressure, has the effect of hastening the separation time. In the case of constant-rate contraction of the body surface, the inertia force is $2|\dot{a}|\dot{\theta}\hat{\theta}$ which acts in the same direction of motion as the particles near the body surface, and therefore has the effect of delaying the separation time. These qualitative conclusions agree with those resulting from the numerical computation of Lin, et. al. for an impulsively starting flow past a cylinder deforming at constant rate.

The effect of non-constant rate of expansion or contraction can also be discussed by adding a term $-\ddot{a}\hat{r}$ to the inertia force. There are four different cases depending on the combination of the signs of $\dot{a}$ and $\ddot{a}$. It can be shown that in the case when $\dot{a} > 0$ and $\ddot{a} < 0$ flow separation is hastened the most, whereas in the case $\dot{a} < 0$ and $\ddot{a} > 0$ flow separation is delayed the most.

If the flow is not started impulsively but rather, is started from rest, accelerating constantly over a time interval τ_c followed by a constant velocity, the effect on the motion of the fluid particles as viewed from the body surface-fixed frame of reference is equivalent to adding an inertia force $\vec{U}$ over τ_c. Such a force is a favorable one and tends to delay flow separation. Therefore, with the same constant $\vec{U}$, larger τ_c will delay flow separation further. This conclusion is also in agreement with that resulting from the numerical computation of Lin et. al. (1986).

§ 6. CONCLUDING REMARKS

In this paper a concise mathematical framework is constructed to study the topology of steady three-dimensional separated flow of an incompressible viscous fluid. With flow separation defined by the existence of a stream surface which intersects with the body surface, it is shown that the line of separation is itself a skin-friction line. Flow separation is classified as being either regular or singular, depending respectively on whether the line of separation contains only a finite number of singular points or is a singular line of the skin-friction field.

In regular separation a line of separation originates from a saddle point of separation of the skin-friction field and ends at nodal points of separation. It is also shown that a saddle point of the skin-friction field on the line of separation is simultaneously a half-nodal point of the flow field on the separation stream surface. Conversely, a nodal point of the skin-friction field on the line of separation is simultaneously a half-saddle point of the flow field on the separation stream surface.

The same mathematical framework proves useful for a study of the topology of unsteady three-dimensional incompressible flow separation when the flow is defined relative to a coordinate system fixed to the body surface. It is shown that separation of an unsteady incompressible viscous flow at time t, when viewed from such a frame of reference, is topologically the same as that of the fictitious steady flow obtained by freezing the unsteady flow at the instant t. Several applications of this result showing effects of various forms of flow unsteadiness on flow separation are discussed qualitatively.

Finally, extension of the results for steady three-dimensional incompressible flow separation to the case of steady compressible flow is straightforward. In the latter case we still have

$$\text{(i)} \quad \vec{V} \text{ is analytic} \tag{58}$$

$$\text{(ii)} \quad \vec{V} = 0 \text{ on the body surface } B \tag{59}$$

But, instead of (2), the continuity equation now reads

$$\nabla \cdot (\rho \vec{V}) = \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0 \tag{60}$$

where ρ is density of the fluid. However, by using (59), equation (60) reduces to

$$(\nabla \cdot \vec{V})_B = 0 \tag{61}$$

which, after using (59) once again, in turn implies that

$$\left(\frac{\partial w}{\partial \varsigma} \right)_{\varsigma = 0} = 0 \tag{62}$$

Equations (58), (59) and (62) for a compressible flow are seen to be identical to (1), (3) and (7) for an incompressible flow. It has been noted earlier (section 2.1) that all topological properties of steady three-dimensional flow separation of an incompressible fluid are derived solely on the basis of equations (1), (3) and (7). As these latter equations are also shared by a compressible fluid, it is concluded that the topology of separation of a steady three-dimensional compressible flow is identical to that of an incompressible flow. We remark, however, that the topologies will not be identical in the case of unsteady flow, in view of the additional term $\partial \rho / \partial t$ that will appear in the continuity equation for compressible unsteady flow.

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